Spectral sets and derivatives of the psd cone

Mario Kummer
TU Berlin

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A spectrahedral cone is a set of the form

\[ S = \{ x \in \mathbb{R}^n : A(x) = x_1 A_1 + \ldots + x_n A_n \text{ is positive semidefinite} \}, \]

where \( A_1, \ldots, A_n \in \text{Sym}_2(\mathbb{R}^d) \) are real symmetric \( d \times d \) matrices.
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- Feasible sets of semidefinite programming.
- Polyhedral cones: Take $A(x)$ to be diagonal.

**Question**

- Which sets $K \subset \mathbb{R}^n$ are spectrahedral?
Spectrahedral cones

\[ S = \{ x \in \mathbb{R}^n : A(x) = x_1A_1 + \ldots + x_nA_n \text{ is positive semidefinite} \} . \]

- Fix \( e \in \text{int}(S) \). W.l.o.g. \( A(e) = I_d \).
- The polynomial \( \det A(x) \) is hyperbolic in the following sense:
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**Definition** A homogeneous polynomial \( h \in \mathbb{R}[x_1, \ldots, x_n] \) is hyperbolic with respect to \( e \in \mathbb{R}^n \) if \( h(e) \neq 0 \) and if \( h(te - v) \) has only real roots for all \( v \in \mathbb{R}^n \).
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\[ C(h, e) = \{ v \in \mathbb{R}^n : h(te - v) \text{ has only nonnegative roots} \}. \]
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- \( \det A(te - v) = \det(tI_d - A(v)) \).
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- \( \det A(te - v) = \det(tI_d - A(v)) \).
- \( S = C(\det A(x), e) \).
Conjecture. Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. Then $C(h, e)$ is spectrahedral.
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True if:

- $\deg h \leq 2$.
- $n \leq 3$. (Helton–Vinnikov)
- $n = 4$ and $\deg h = 3$. (Buckley–Košir)
The following polynomials are hyperbolic with respect to $e$:

- $\det A(x)$ for $A(x)$ real symmetric matrix with linear entries and $A(e)$ positive definite.

Includes spanning tree polynomials of graphs, bases generating polynomials of regular matroids and ternary hyperbolic polynomials.
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Includes spanning tree polynomials of graphs, bases generating polynomials of regular matroids and ternary hyperbolic polynomials.

Their hyperbolicity cones are clearly spectrahedral.
The following polynomials are hyperbolic with respect to $e$:

- The homogeneous multivariate matching polynomial of an undirected graph $G = (V, E)$:

$$\mu_G(x, w) = \sum (-1)^{|M|} \cdot \prod_{v \notin V(M)} x_v \cdot \prod_{e \in M} w_e^2$$

where the sum is over all matchings $M$ of $G$. (Heilmann–Lieb)

- $e = (1_V, 0_E)$. 
Constructing hyperbolic polynomials

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Their hyperbolicity cones are spectrahedral (Amini).
The following polynomials are hyperbolic with respect to $e$:

- The defining polynomial of the $k$th secant variety of a projectively normal $M$-curve with “many” pseudolines in $\mathbb{P}^{2k+2}$. (K.–Sinn)

Their hyperbolicity cones are spectrahedral for rational and elliptic curves.
These operations preserve being hyperbolic with respect to $e$:

- Taking products.
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These operations also preserve spectrahedrality of the corresponding hyperbolicity cones.
Consequence of Rolle’s Theorem:

- If a polynomial \( p \in \mathbb{R}[t] \) has only real zeros, then its derivative \( p' \) has only real zeros.
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This implies:

- If a polynomial \( h \in \mathbb{R}[x_1, \ldots, x_n] \) is hyperbolic with respect to \( e \), then its directional derivative

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D_e h = \sum_{i=1}^{n} e_i \cdot \frac{\partial h}{\partial x_i}
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**Question** Is the hyperbolicity cone of \( D_\mathbf{e}(\det A(x)) \) spectrahedral?
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This implies:

- If a polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is hyperbolic with respect to $e$, then its directional derivative

$$D^k_e h = \sum_{i=1}^n e_i \cdot \frac{\partial h}{\partial x_i}$$

is hyperbolic with respect to $e$ as well.

**Question** Is the hyperbolicity cone of $D^k_e (\det A(x))$ spectrahedral?
The polynomial

\[ h = x_1 \cdots x_d \]

is hyperbolic with respect to \( e = (1, \ldots, 1) \).
Example

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\[ \text{D}_{e}^{n-k} h = (n - k)! \sigma_{k,d} \]

where \( \sigma_{k,d} \) is the elementary symmetric polynomial in \( d \) variables of degree \( k \).
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- \( \sigma_{k,d} \) is hyperbolic with respect to \( e = (1, \ldots, 1) \).
- The hyperbolicity cone of \( \sigma_{k,d} \) is spectrahedral (Brändén).
**Renegar derivatives**

**Question** Is the hyperbolicity cone of $D^k_e(\det A(x))$ spectrahedral?

- It suffices to prove that the hyperbolicity cone of $D^k_I(\det X)$ is spectrahedral where $X$ is the *generic* $d \times d$ symmetric matrix and $I$ the $d \times d$ identity matrix.
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- It suffices to prove that the hyperbolicity cone of $D^k_{I}(\det X)$ is spectrahedral where $X$ is the generic $d \times d$ symmetric matrix and $I$ the $d \times d$ identity matrix.

Let us write

$$\det(tI - X) = \sum_{k=0}^{d} (-1)^k p_k t^{d-k}$$

for suitable polynomials $p_k$ of degree $k$ ($p_1 = \text{tr}(X), p_d = \det(X)$).
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- $p_k = \frac{1}{(d-k)!} D^k_I(\det X)$. 
Question Is the hyperbolicity cone of $D^k_e(\det A(x))$ spectrahedral?

- It suffices to prove that the hyperbolicity cone of $D^k_l(\det X)$ is spectrahedral where $X$ is the generic $d \times d$ symmetric matrix and $I$ the $d \times d$ identity matrix.

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$$\det(tI - X) = \sum_{k=0}^{d} (-1)^k p_k t^{d-k}$$

for suitable polynomials $p_k$ of degree $k$ ($p_1 = \text{tr}(X)$, $p_d = \det(X)$).

- $p_k = \frac{1}{(d-k)!} D^d_{l-k}(\det X)$.
- $p_k = \sigma_{k,d}(\lambda(X))$ where $\sigma_{k,d}$ is the elementary symmetric polynomial of degree $k$ in $d$ variables and $\lambda(X)$ the vector of eigenvalues of $X$. 
Theorem (Bauschke–Güler–Lewis–Sendov) Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ a symmetric polynomial that is hyperbolic with respect to $e = (1, \ldots, 1)$. Consider the function

$$H : \text{Sym}_2(\mathbb{R}^d) \to \mathbb{R}, \; X \mapsto h(\lambda(X))$$

where $\lambda(X)$ is the vector of eigenvalues of $X$.

a) $H$ is a polynomial.
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where $\lambda(X)$ is the vector of eigenvalues of $X$.

a) $H$ is a polynomial.

b) $H$ is hyperbolic with respect to $I$.

c) $C(H, I) = \{X : \lambda(X) \in C(h, e)\}$.
Definition (Sanyal–Saunderson) A spectral convex set is a set of the form \( \{ X \in \text{Sym}_2(\mathbb{R}^d) : \lambda(X) \in K \} \) for some symmetric convex set \( K \subset \mathbb{R}^d \).

- Raman’s talk on Thursday!
Corollary

A symmetric $d \times d$ matrix $A$ is in the hyperbolicity cone of $D_I^{d-k}(\det X)$ if and only if its spectrum $\lambda(A)$ is in the hyperbolicity cone of the elementary symmetric polynomial $\sigma_{k,d}$. 
Corollary

A symmetric $d \times d$ matrix $A$ is in the hyperbolicity cone of $D_i^{d-k}(\det X)$ if and only if its spectrum $\lambda(A)$ is in the hyperbolicity cone of the elementary symmetric polynomial $\sigma_{k,d}$.

Using this and a spectrahedral representation of the hyperbolicity cone of $\sigma_{d-1,d}$ due to Sanyal, Saunderson proved that the hyperbolicity cone of $D_i^1(\det X)$ is spectrahedral.
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- Using this and a spectrahedral representation of the hyperbolicity cone of $\sigma_{d-1,d}$ due to Sanyal, Saunderson proved that the hyperbolicity cone of $D_i^1(\det X)$ is spectrahedral.
- Brändén constructed a spectrahedral representation of the hyperbolicity cone of $\sigma_{k,d}$ for all $k$. 
**Question** Let $S \subseteq \mathbb{R}^n$ be a spectrahedral cone which is symmetric under permuting the coordinates. Is the spectral convex set

$$\Lambda(S) = \{ A \in \text{Sym}_2(\mathbb{R}^n) : \lambda(A) \in S \}$$

also spectrahedral?
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- $\Lambda(S)$ is a hyperbolicity cone. (Bauschke–Güler–Lewis–Sendov)
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also spectrahedral?

- $\Lambda(S)$ is a hyperbolicity cone. (Bauschke–Güler–Lewis–Sendov)
- Yes, if $S$ is a polyhedral cone. (Sanyal–Saunderson)
Definition

A representation of $\mathfrak{S}_n$ is *short* if it consists only of such irreducible representations that correspond to partitions of length at most 2.
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Short:

Not short:
Example
Let $\mathcal{M}_{d,n} \subset \mathbb{R}[x_1, \ldots, x_n]$ be the vector space of all homogeneous multiaffine polynomials of degree $d$. Then $\mathcal{M}_{d,n}$ is a short representation:

- $\mathcal{M}_{d,n} = \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}} (\text{Trv})$

- Young’s rule: $\mathcal{M}_{d,n} = \bigoplus_{i=0}^{\min(d,n-d)} V_{n-i,i}$
The main result

**Theorem**  
Let $V$ be a short representation of $\mathfrak{S}_n$ and $\varphi : \mathbb{R}^n \to \text{Sym}_2(V)$ an $\mathfrak{S}_n$-linear map. Let $S \subset \mathbb{R}^n$ be the preimage of the positive semidefinite cone in $\text{Sym}_2(V)$ under $\varphi$. Then $\Lambda(S) \subset \text{Sym}_2(\mathbb{R}^n)$ is a spectrahedral cone.
**The main result**

**Theorem**

Let $V$ be a short representation of $S_n$ and $\varphi : \mathbb{R}^n \to \text{Sym}_2(V)$ an $S_n$-linear map. Let $S \subset \mathbb{R}^n$ be the preimage of the positive semidefinite cone in $\text{Sym}_2(V)$ under $\varphi$. Then $\Lambda(S) \subset \text{Sym}_2(\mathbb{R}^n)$ is a spectrahedral cone.

**Corollary**

The hyperbolicity cone of $D^k_I(\det A(x))$ spectrahedral.

- For any fixed $k$, the size of this spectrahedral representation is $O(n^2 \cdot (\min(k, n-k)+1))$ when the size $n$ of $A(x)$ grows.
Theorem Let $V$ be a short representation of $\mathfrak{S}_n$ and

$$\varphi : \mathbb{R}^n \to \text{Sym}_2(V)$$

an $\mathfrak{S}_n$-linear map. Then there is a representation $W$ of $O(n)$ and an $O(n)$-linear map map

$$\Phi : \text{Sym}_2(\mathbb{R}^n) \to \text{Sym}_2(W)$$

such that $\Phi(A)$ is positive semidefinite if and only $\varphi(\lambda(A))$ is positive semidefinite.
Idea of the proof

Let \( 0 \leq 2d \leq n \). We have \( \mathcal{M}_{d,n} = \bigoplus_{i=0}^{d} V_{n-i,i} \). More precisely:

- \( V_{n-i,i} = \ker(D^{d-i+1}_e) \cap \ker(D^d_e)^\perp \)
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Let $\text{Min}_{d,n}$ the vector space spanned by the $d \times d$ minors of the generic symmetric $n \times n$ matrix.
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The decomposition of the $O(n)$-module $\text{Min}_{d,n}$ into irreducibles is $\text{Min}_{d,n} = \bigoplus_{i=0}^{d} E^{(i,i)'}$. 
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- Here $E^{(i,i)} = \ker(D_{I}^{d-i+1}) \cap \ker(D_{I}^{d-i})^\perp$. 
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$\triangleright \ $ The decomposition of the $O(n)$-module $\text{Min}_{d,n}$ into irreducibles is $\text{Min}_{d,n} = \bigoplus_{i=0}^{d} E(i,i)'$.

$\triangleright \ $ Here $E(i,i)' = \ker(D_{f}^{d-i+1}) \cap \ker(D_{f}^{d-i})^{\perp}$.

To obtain $\mathcal{W}$ replace each $V_{n-i,i}$ in $V$ by $E(i,i)'$. 
Theorem (Newton) The function

\[ N_k : \text{Sym}_2(\mathbb{R}^n) \to \mathbb{R}, \]

\[ X \mapsto (k(n - k)\sigma_{k,n}^2 - (k + 1)(n - k + 1)\sigma_{k-1,n} \cdot \sigma_{k+1,n})(\lambda(X)) \]

is nonnegative.
Newton’s inequalities and sums of squares

**Theorem (Newton)**  The function

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\[ X \mapsto (k(n-k)\sigma_{k,n}^2 - (k+1)(n-k+1)\sigma_{k-1,n} \cdot \sigma_{k+1,n})(\lambda(X)) \]

is nonnegative.

**Theorem**  The function \( N_k \) is a sum of squares of polynomials (in the entries of \( X \)).
Thanks!